Classical Mechanics
ISI B.Math
Backpaper Exam : December 29, 2022

Total Marks: 50
Time : 3 hours

Answer all questions:

1. (**Marks = 7 + 2 + 1**)

An electron of mass m and charge $-e$ is moving under the combined influence of a uniform electric field $E_0\mathbf{j}$ and a uniform magnetic field $B_0\mathbf{k}$. [Recall that the Lorentz force on a particle with charge q is given by $\mathbf{F} = q\mathbf{E} + q(\mathbf{v} \times \mathbf{B})$] . Initially the electron is at the origin and is moving with velocity $u\mathbf{i}$.
(a) Show that the trajectory of the electron is given by

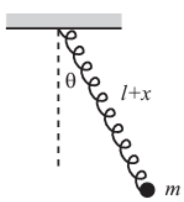
$$x = a(\Omega t) + b \sin \Omega t,\ y = b(1 - \cos \Omega t),\ z = 0$$

where $\Omega = \frac{eB_0}{m}$, $a = \frac{E_0}{\Omega B_0}$ and $b = \frac{(uB_0 - E_0)}{\Omega B_0}$.

(b) Is the energy of the particle described in part (a) conserved ? Is the angular momentum about the origin conserved ? Justify your answers. Now consider the same system with the electric field turned off. Does this change the answers to the previous two questions ?

(c) In part (a) if the electric field is turned off and initially the electron is at the origin moving with a velocity $u\mathbf{i} + w\mathbf{k}$ how will the trajectory in (a) get modified ?

2. (**Marks = 2 + 3 + 5**)



Consider an "elastic pendulum": A particle of mass m is attached to an elastic string of stiffness K and unstretched length l. The spring is arranged to lie in a straight line (which we can arrange by, say, wrapping the spring around a massless rod). Assume that the mass moves in a vertical plane.

(a) Write down the Lagrangian of the system choosing appropriate generalized coordinates . Are there any cyclic coordinates ? Is the total energy for the system conserved ?

(b) Find the Lagrange equations for this system from (a) .

(c) Solve the Lagrange equations in the approximation of small angular (θ) and radial displacements(x) from equilibrium and corresponding small velocities.
Your solution may contain arbitrary constants to be determined from initial conditions.

3. (**Marks = 4 + 6**)

(a) A particle of mass m moves in the central force field $\mathbf{F} = -(\frac{m\gamma}{r^2})\hat{\mathbf{r}}$, where γ is a positive constant. Show that bounded and unbounded orbits are possible depending on the value of the total energy E

(b) An asteroid is approaching the sun from a great distance. At this time it has a constant speed u and is moving in a straight line whose perpendicular distance from the sun is p. For the special case in which $u^2 = \frac{4M_S G}{3p}$ (where M_S is the mass of the Sun, and G the gravitational constant), find the distance of closest approach of the asteroid to the Sun and the speed of the asteroid at the time of closest approach.

4. (**Marks = 2 + 4 + 2 + 2**)

Consider the motion of a particle of mass m under the influence of a force $\mathbf{F} = -k\mathbf{r}$ where k is a positive constant and $\mathbf{r}$ is the position vector of the particle.

(a) Show that the motion of the particle lies in a plane.

(b) Find the position of the particle as a function of time, assuming that at $t = 0, x = a, y = 0$ and $v_x = 0, v_y = v_0$.

(c) Show that the orbit is an ellipse.

(d) Find the period of motion.

5. (**Marks = 5 + 5**)

(a) A uniform ball of mass M and radius a is pivoted so that it can turn freely about one of its diameters which is fixed in a vertical position. A beetle of mass m can crawl on the surface of the ball. Initially the ball is rotating with angular speed Ω with the beetle at the north pole. The beetle then walks (in any manner) to the equator of the ball and sits down. What is the angular speed of the ball now ?

(b) Consider a thin homogeneous plate that lies in the $x_1 - x_2$ plane.
Show that the inertia tensor takes the form, where A, B, C are constants

$$\{\mathbf{I}\} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$